1.0

4.5
5.0
5.6
6.3

2.8  2.5

3.2  2.2

3.6

4.0  2.0

1.1

1.8

1.25  1.4  1.6

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Ladder forms are probably the most promising canonical forms in estimation and system identification. Many record applications, such as in geophysical signal processing, high resolution ("maximum entropy") spectral estimation and speech encoding, justify the interest in these forms. They appear in many contexts, such as scattering and network theory and the theory of orthogonal polynomials. The state-space model ladder realizations are very closely related in (block) Schwarz matrix canonical forms, which generally appear in the context of stability analysis. In fact they are the natural "stability canonical form"

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for (discrete-time) Lyapunov equations since the associated positive definite (covariance) matrices are diagonal resp. an identity. This fact leads also to close connections to square-root algorithms including the ones of Cholesky and Chandrasekhar type, since again Ladder forms are the natural canonical forms. In realization theory these forms are obtained via orthonormal state-space bases using Gram-Schmidt type procedures. Ladder forms have many other advantages, such as lowest computational complexity, good numerical behavior, stability "by inspection" properties and relations to physical properties such as reflection or partial correlation coefficients, and perhaps absorption coefficients.

We shall present an outline of some newer results connecting these topics and present new examples of our new exact least-squares recursions for ("adaptive") ladder forms with poles and zeros. We close with a few simulation examples, including the identification of a layered media (via ultra-sound).

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## Abstract

Ladder forms are probably the most promising canonical forms in estimation, and system identification. Many recent applications, such as in geophysical signal processing, high resolution ("maximum entropy") spectral estimation and speech encoding, justify the interest in these forms. They appear in many contexts, such as scattering and network theory and the theory of orthogonal polynomials. The state-space model ladder realizations are very closely related to (block) Schwarz matrix canonical forms, which generally appear in the context of stability analysis. In fact they are the natural "stability canonical form" for (discrete-time) Lyapunov equations since the associated positive definite (covariance) matrices are diagonal resp. an Identity. This fact leads also to close connections to square-root algorithms including the ones of Cholesky and Chandrasekhar type, since again ladder forms are the natural canonical forms. In realization theory these forms are obtained via orthonormal state-space bases using Gram-Schmidt type procedures. Ladder forms have many other advantages, such as lowest computational complexity, good numerical behavior, stability "by inspection" properties and relations to physical properties such as reflection or partial correlation coefficients, and perhaps absorption coefficients.

We shall present an outline of some newer results connecting these topics and present new examples of our new exact least-squares recursions for ("adaptive") ladder forms with poles and zeros. We close with a few simulation examples, including the identification of a layered media (via ultra-sound).

## 1. Introduction

Ladder forms have attracted much attention recently because they are probably the most promising canonical forms in estimation and system identification. These forms have appeared in many applications such as geophysical signal processing for quite some time , and more recently such models are being used in high resolution ("maximum entropy") spectral estimation and speech encoding. Ladder ( sometimes called lattice - a term we would like to reserve for two and higher dimensional extensions [LKM]) forms appear in many contexts, first perhaps in scattering and network theory where the scattering of waves in layered media or in (non-homogeneous) transmission lines leads very naturally to ladder forms, see e.g. [Gla2], [LKF], [RMT], [Kelly].

Ladder forms appear explicitly but more often implicitly in many contexts. They are directly related to the scattering of waves and therefore perhaps first introduced in physics. Some of the associated mathematics are used in network theory, where the cascade structure of the ladder forms plays an important role. The notion of transfer functions leads very naturally to the next connection, the theory of orthogonal polynomials. They in turn also appear in the stability analysis of linear systems. The state-space models that are related to orthogonal (matrix) polynomials are the so-called (block) Schwarz matrix canonical forms, see e.g. [AJM], [SS]. However, the special structure of these matrices leads very

directly to the ladder realizations [Mo]. In fact they are the natural "stability canonical form" for (discrete-time) Lyapunov equations, since the associated positive definite (covariance) matrices are diagonal or respectively an identity matrix. The similarity transformations to this form involve a matrix square-root of the associated covariance matrix. The ladder forms are therefore closely connected to square-root algorithms including the ones of Cholesky and Chandrasekhar type. In realization theory these forms are obtained via orthonormal state-space bases using Gram-Schmidt type procedures, due to the fact that this ortho-normalization is again related to matrix square-root and orthogonal polynomials.

Ladder forms have many other interesting properties. Due to the fact that they are in many problems the "natural canonical form", they lead to algorithms with lowest computational complexity compared to other canonical forms. Although a detailed study is still outstanding, there are many indications that this form leads to good numerical behavior of the associated algorithms, a property that is not shared with most canonical forms. Furthermore, the stability "by inspection" property given the ladder coefficients is shared only by the Jordan or modal canonical form. However, the latter one requires the knowledge of the eigen-values that are in general not very easily obtained, compared to the finite algorithm required to get the ladder coefficients. They in turn have other interesting interpretations and relations to physical properties such as reflection, and perhaps absorption coefficients. In stochastic process modeling and spectral estimation the ladder coefficients turn out to be partial correlation or canonical correlation coefficients, which leads to very simple methods to determine these parameters either from covariance data or even directly from measured data.

In [MLNV] we presented a classification of exact least-squares modeling methods. The material discussed here is a sequel to the results discussed there, in particular we will concentrate here on the ladder forms and the associated algorithms. We shall present an outline of some newer results connecting these topics and

present new examples of our new exact least-squares recursions for ("adaptive") ladder forms with poles and zeros. We close with a few simulation examples, including the identification of a layered media (via ultra-sound).

## II. Ladder Realizations

In [MLNV] and [MVL] we discussed various ladder realizations. We assume here familiarity with this material and would like to give here only a missing link to state space realizations, namely the fact that the ladder forms can be obtained via an ortho-normalization of the state space. In this context it is well known, that various canonical state space realization can be obtained via methods that construct a basis of either the Hankel matrix of the Markov parameters, resp. the impulse response parameters of the system, or bases of the controllability or observability matrices of the system, see e.g. [K-S74]. We will present here an outline of the scalar discrete-time constant parameter case. For convenience we use an intermediate canonical form, the controller form. It has the property that the $i^{th}$ component of the $n$ state vector $x^i(z)$ can be obtained from the

input $u(z)$ in Z-transform notation via $x^i(z) = u(z)z^{n-i}/a(z)$, where $a(z)$ is the characteristic polynomial. The ladder forms are obtained in a similar manner via $x_i{}^i(z) = u(z)b^i(z)/a(z)$, where $b^i(z)$ is the $i^{th}$ (dual) ortho-normal polynomial on the unit circle [Sze]. Writing these facts in matrix notation, we obtain the results that the the state at time $n$ is given by $x_n = C_n u[n-1,0]$, $u(i,j) = [u_i, ..., u_j]$, where $C_n$ is the usual controllability matrix. For the controller form it is now not too surprising that $C_n{}^c = UT(a_1,...,a_n)^{-1}$, the inverse of a unit upper triangular Toeplitz matrix of the coefficients of $a(z)$. the Ladder forms on the other hand result in (see [Mo]) $C_n{}^l = B_n UT(a_1,...,a_n)^{-1}$, where $B_n$ is a lower triangular matrix containing as rows the coefficient vectors of $b^i(z)$. Due to its orthogonality property $B_n B_n{}' = R_n$, the $n$ by $n$ Toeplitz matrix associated with the (Z-transformed) correlation function $R(z) = 1/|a(z)|^2$. $R_n$ is also the steady state covariance matrix of the controller form $R_n{}^c$, if $u(z)$ the input is a white process. Now, since the similarity transform matrix $S$ from one state space form to an other is given for instance by the ratio of the controllability matrices, it is clear that $S = C^l (C^c)^{-1} = B_n = (R_n{}^c)^{-1/2}$ i.e. from an arbitrary (controllable) state space form the similarity transform is given by the inverse square-root of the steady state covariance. This leads finally to the connection with Lyapunov equation type characterization of the ladder forms, namely that their covariances are an Identity (or diagonal) which is precisely the characteristic property of Schwarz matrizes, see [AJM], the state space feedback matrix of ladder forms [Mo]. Due to limitations we defer a more detailed discussion of the details and various extensions to [ML].

## III. LS Recursions for Ladder Forms
### The Prewindowing Case

In [MLVK] we presented this case, for completeness and in order to correct some typographical errors we repeat some of the equations here. Given a series of observations $\{y(t), 0 \le t \le T\}$, where $\{y(\cdot)\}$ can be $m$ vectors, we wish to find the least-squares one-step predictor of order $p$ parametrized by the (matrix) coefficients $\{A_{p,T}(i), i=1,...,p\}$. We can define many different squared error criteria $E_{p,T}$, for instance as a function of $s$ and $f$ in

$$E_{p,T} \hat{=} \sum_{t=s}^{f} \epsilon'_{p,T}(t)\epsilon_{p,T}(t), \quad \epsilon_{p,t} \hat{=} A'_{p,T} y[t:t-p],$$

$$A'_{p,T} \hat{=} [I, A'_{p,T}(1),...,A'_{p,T}(p)], \quad y'[t:t-p] \hat{=} [y_t', ..., y_{t-p}'] \quad (\text{III-1})$$

An obvious choice from an *innovations* point-of-view is ($s=0, f=T$), the "pre-windowing" case [MDKV]. If $s = p$ and $f = T$ the so-called "covariance" method is obtained [MDKV], [MVL], and if $s = 0$ and $f = T + p$ we get the pre- and post-windowed case or the "correlation" method [MC]. The total squared error can be expressed as

$$E_{p,T} = tr\{A'_{p,T} R_{p,T} A_{p,T}\}, \quad R_{p,T} = Y_{p,T} Y'_{p,T},$$

$$Y_{p,T} \hat{=} [y[0:-p], y[1:-p+1],...,y[T:T-p]] \quad (\text{III-2})$$

Thus the problem of determining $A_{p,T}$ by minimizing $E_{p,T}$ leads to

$$R_{p,T} A_{p,T} = [R^\epsilon_{p,T}, 0, ..., 0]', \quad tr R^\epsilon_{p,T} = min E_{p,T} \quad (\text{III-3})$$

Although $R_{p,T}$ is not Toeplitz, it still carries a certain shift-invariance structure, given by the following identities

$$R_{p,T} = R_{p,T-1} + y[T:T-p] y[T:T-p]' \quad (\text{III-4})$$

$$= \begin{bmatrix} x & x & x \\ x & R_{p-1,T-1} & x \end{bmatrix} = \begin{bmatrix} R_{p-1,T} & x \\ x & x & x \end{bmatrix}. \quad (\text{III-5})$$

Define the backward predictor $B_{p,T}$ and the smoothing errors $C_{p,T}$

$$B'_{p,T} R_{p,T} \hat{=} [0,...,0, R^r_{p,T}]; \quad C'_{p,T} R_{p,T} \hat{=} y[T:T-p] \quad (\text{III-6})$$

Then the forward and backward prediction errors (innovations), $\epsilon_{p,T}$, and $r_{p,T}$, and an auxiliary scalar $\gamma_{p,T}$ can be defined by

$$[\epsilon'_{p,T}, r'_{p,T}, \gamma_{p,T}] \hat{=} y'[T:T-p][A_{p,T}, B_{p,T}, C_{p,T}]$$

### Order Update Recursions

Using the three shift-invariance identities for $R_{p,T}$ (III-5) and using some symmetry properties, the order update recursions for $A_{p,T}, B_{p,T}, C_{p,T}, R^\epsilon_{p,T}$, and $R^r_{p,T}$ are

$$A'_{p+1,T} = [A_{p,T} 0]' - \Delta'_{p+1,T} R^{-r}_{p,T-1} [0, B'_{p,T-1}]'$$

$$B'_{p+1,T} = [0, B_{p,T-1}]' - \Delta_{p+1,T} R^{-\epsilon}_{p,T} [A'_{p,T} 0]^T \quad (\text{III-7})$$

$$C'_{p+1,T} = [C_{p,T} 0]' + r'_{p+1,T} R^{-r}_{p+1,T} B'_{p+1,T} \quad \text{where}$$

$$\Delta_{p+1,T} = [\text{last block row of } R_{p+1,T}] [A'_{p,T} 0]'$$
$$= [0, B'_{p,T-1}] [\text{first block row of } R_{p+1,T}]'.$$

$$R^\epsilon_{p+1,T} = R^\epsilon_{p,T} - \Delta'_{p+1,T} R^{-r}_{p,T-1} \Delta_{p+1,T}$$

$$R^r_{p+1,T} = R^r_{p,T} - \Delta_{p+1,T} R^{-\epsilon}_{p,T} \Delta'_{p+1,T}, \quad (\text{III-8})$$

The order update recursions are very similar to the multivariate version of the Levinson algorithm, and a similar set of recursions for time-update can also be obtained [MDKV],[Mo].

### Ladder Type Realization

Premultiplying the above equations by $y'[T:T-p+1]$, we obtain the following order update recursions for $\epsilon_{p,T}, r_{p,T}, \gamma_{p,T}$

$$\epsilon_{p+1,T} = \epsilon_{p,T} - \Delta'_{p+1,T} R^{-r}_{p,T-1} r_{p,T-1}$$

$$r_{p+1,T} = r_{p,T-1} - \Delta_{p+1,T} R^{-\epsilon}_{p,T} \epsilon_{p,T}$$

$$\gamma_{p+1,T} = \gamma_{p,T} + r'_{p+1,T} R^{-r}_{p+1,T} r_{p+1,T}. \quad (\text{III-9})$$

The "Kalman gain" $\Delta_{p+1,T}$ is obtained from (III-5),(III-7) (cf.[MV]) via

$$\Delta_{p+1,T+1} = \Delta_{p+1,T} + r_{p,T} \epsilon'_{p,T+1} / (1 - \gamma_{p-1,T}) \quad (\text{III-10})$$

and the *reflection* or *PARCOR coefficients* are obtained by

$$K^\epsilon_{i+1,T} \hat{=} \Delta_{i+1,T} R^{-\epsilon}_{i,T}; \quad K^r_{i+1,T} \hat{=} \Delta'_{i+1,T} R^{-r}_{i,T} \quad (\text{III-11})$$

The initial conditions are given by

$$\epsilon_{0,T} = r_{0,T} = y_T; \quad \gamma_{-1,T} = 0;$$

$$R^\epsilon_{0,T} = R^r_{0,T} = \sum_{t=0}^{T} y_t y_t' = R^\epsilon_{0,T-1} + y_T y_T';$$

for $p \ge T$:

$$\epsilon_{p,T} = \epsilon_{T,T}; \quad r_{p,T} = r_{T,T}; \quad \gamma_{p,T} = \gamma_{T,T};$$

$$R^\epsilon_{p,T} = R^\epsilon_{T,T}; R^r_{p,T} = R^r_{T,T}; \Delta_{p+1,T} = 0;$$

$$\Delta_{p+1,p+1} = y_0 \epsilon'_{p+1}.$$

As the dual to the stochastic forms in [IS], [Wak], [Mo], [SKM], equations (III-8)-(III-11) are a complete set of *order and time update recursions* to obtain the *exact least-squares* ladder form predictor, which is shown in Figure 1.
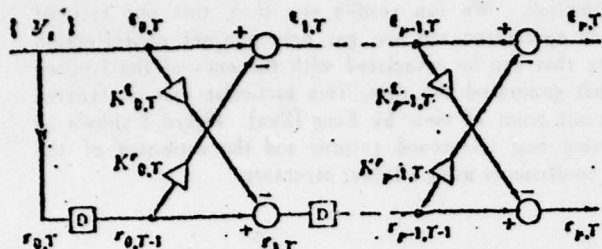
*Figure 1. Ladder realization of exact one-step least-squares predictor.*

The recursion (III-10) computes the *sample cross-covariance* of the *forward and backward innovations*, using the *optimal weighting* $1/(1-\gamma_{..})$, compared to other suboptimal schemes [SV]. See in the appendix a sample comparison of the exact versus two approximate methods. In the scalar case $R_{p,T} > 0$ if $y_0 = 0$, or in general if $\gamma_{p-1,T} < 1$, since $0 \leq \gamma_{p,T} \leq 1$ [MDKV], [MVL]. If $m > 1$, we require $T \geq p \cdot m$. These singularities can be avoided by including a priori estimates of the covariance $R_n$, or equivalently including a weighted norm of the predictor $a_n$ in the error criteria $E_{p,T}$. For tracking of time-varying parameters, e.g. in speech modeling methods, these equations can be modified, either by puting an exponential weight on past errors as discussed in [MKL], (implemented in the simulation in the appendix). Alternatively, the lower bound of the error criterion in (III-1) can be increased, e.g. $s = T - f$, where $f$ is the (constant) "sliding" time frame width of the analysis. This corresponds also to a sliding window on the prediction errors. The resulting equations are similar to the ones in [MVL].

Instead of computing the scalars $\gamma$ one can also work with a second set of prediction errors based on the "old" parameter estimates, since

$$\epsilon_{p,T}(T+1) = \epsilon_{p,T-1} / (1 - \gamma_{p-1,T}) .$$

This alternate form was also found by J. Baker, IEM Yorktown (private communication). A similar situation occurs in the Fast Cholesky (least-squares) algorithms for estimating moving-average parameters via feedback filters described in [Mo], where a "second filter" or "predictor filter" appears that computes variables of the type $\epsilon_{p,T}(T+1)$. It is interesting to note in this context, that the unwindowed ("covariance") method actually also leads to *signal feedback* paths (actually a smoothing filter), see [MVL], but the simpler prewindowing case is feed forward only.

Many modifications have been proven useful in actual implementations, they are partially due to the fact that many additional identities exist and others are due to differences in numerical behavior and trade-offs in operations count and memory requirements. Systematic experiments are now in progress and will be reported on shortly.

## IV. LS-Recursions for Rational Ladder Forms
### Rational or ARMA Modeling

Rational or pole-zero or ARMA modeling methods were described in [SLC], [MKL] and their relation to joint innovations representation via an imbedding of the ARMA model in a two (m) channel AR model in [MLNV] and [Mo]. The same idea also leads to stable partial minimal realizations of the joint impulse-response and covariance-matching type [MLNV]. Given an ARMA model as represented by the difference equation we can rewrite it as

$$y_t + a_1 y_{t-1} + \ldots + a_n y_{t-n} - b_1 u_{t-1} - \ldots - b_n u_{t-n} = b_0 u_t , \quad \text{(IV-1)}$$

or $a'y_t - b_1'u_t = b_0 u_t$, where

$$a' = [1, a_1, \ldots, a_n], \quad y_t' = [y_t, \ldots, y_{t-n}],$$
$$b_1' = [0, b_1, \ldots, b_n], \quad u_t' = [u_t, \ldots, y_{t-n}] .$$

Now consider the following augmented equation

$$\begin{bmatrix} a' & -b_1' \\ 0 & e_1' \end{bmatrix} \begin{bmatrix} y_t \\ u_t \end{bmatrix} = \begin{bmatrix} b_0 u_t \\ u_t \end{bmatrix}, \quad \text{(IV-2)}$$

($e_1'$ is the first unit vector). This equation can be interpreted as an AR model for the joint process $\{y, u\}$ [Mo], since the RHS is equal to the joint innovations of $\{y, u\}$, since

$$\epsilon_t = \begin{bmatrix} \epsilon^y_t \\ \epsilon^u_t \end{bmatrix} = \begin{bmatrix} y_t - \hat{y}_{t|t-1} \\ u_t - \hat{u}_{t|t-1} \end{bmatrix} = \begin{bmatrix} b_0 u_t \\ u_t \end{bmatrix}. \quad \text{(IV-3)}$$

### Stochastic Case

From a stochastic process point of view we can express the normal equation associated with the augmented AR model as

$$E\left\{ \begin{bmatrix} y_t \\ u_t \end{bmatrix} [y_t' \, u_t'] \right\} \begin{bmatrix} a & 0 \\ -b_1 & e_1 \end{bmatrix} = E\left\{ \begin{bmatrix} y_t \\ u_t \end{bmatrix} [u_t b_0 \, u_t] \right\}$$

$$= \begin{bmatrix} R_n & T_n' \\ T_n & I_n \end{bmatrix} \begin{bmatrix} a & 0 \\ -b_1 & e_1 \end{bmatrix} = \begin{bmatrix} e_1 b_0 b_0' & e_1 b_0 \\ e_1 b_0 & e_1 \end{bmatrix}. \quad \text{(IV-4)}$$

We can solve for the normal equation of $a_n$:

$$R_n a_n = [R_n - T_n' T_n] a_n = [H_\infty' H_\infty] a_n = e_1 R^\epsilon_n \text{(IV-5)}$$

The equations (IV-4) and therefore the non-Töplitz equations (IV-5) (!) can be solved recursively with the LWR algorithm. Note that if $R^\epsilon_k = 0$, the minimal order $n = k$. We could bring equations (IV-4) into a more familiar form by the interleaving permutation $(1,3,5,\ldots,2n-1,2,4,6,\ldots,2n+2)$, cf. [MDKV], to convert the two-process covariance matrix into a n by n block Töplitz matrix, with 2 by 2 blocks, however the LWR algorithm clearly applies to both representations with suitable modifications.

Thus we have shown that the joint impulse response & covariance matching problem is equivalent to solving a set of normal equations associated with a two channel AR modeling problem. Since the predictor for the joint process is *triangular* and *minimum phase*, the denominator $a_n$ of the underlying ARMA model is also *minimum phase* and therefore *stable*, (for all $k$).

Equations (IV-4) and the elegant stability proof were actually first obtained by Claerbout [Cla1] via a least-squares rational approximation. The connections between the joint innovations representation, the augmented normal equations, and the Hankel matrix were pointed out in [Mo] and also in [MDKV], [MKD], [DKM], where algorithms were given to solve equations of the type seen in (IV-4) and (IV-5).

### Deterministic Case

In [MLNV] we considered the deterministic case where we are given impulse response data or the Markov parameters, here we shall assume that we are given a series of observations and we want to find a least-squares (deterministic) one-step ARMA predictor recursively from the data equivalent to the RML algorithms described in [SLC] and [MKL]. Our approach will not give a new way how to derive these algorithms, but it will also give us very quickly the ladder forms.

Writing the input/output relationship in matrix notation yields

$$\begin{bmatrix} T_{n,T} & 0 \\ \Pi_T & T_T \end{bmatrix} \begin{bmatrix} a_{n,T} \\ 0_T \end{bmatrix} = \begin{bmatrix} b_{n,T} \\ 0_T \end{bmatrix}, \quad \text{(IV-5)}$$

where $T_{n,T}$ lower-triangular and $\Pi_T$ is a full matrix, but both are a product of two Töplitz matrices containing the data and the normalized one-step prediction errors, which take place of the inputs $u = R^{-\epsilon/2} \epsilon$, $\epsilon = y_t - \hat{y}_{t|t-1}$, where $R^\epsilon_{n,T} = E_{n,T}$ in

$$\epsilon_t = \begin{bmatrix} \epsilon^y_t \\ \epsilon^u_t \end{bmatrix} = \begin{bmatrix} y_t - \hat{y}_{t|t-1} \\ u_t - \hat{u}_{t|t-1} \end{bmatrix} = \begin{bmatrix} b_0 u_t \\ u_t \end{bmatrix}.$$

$$\begin{bmatrix} T_{n,T}{}' & \Pi_T{}' \\ I & 0 \end{bmatrix} \begin{bmatrix} T_{n,T} & I \\ \Pi_T & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ -b_1 & e_1 \end{bmatrix} =$$

$$= \begin{bmatrix} R_n & T_{n,T}{}' \\ T_{n,T} & I \end{bmatrix} \begin{bmatrix} a & 0 \\ -b_1 & e_1 \end{bmatrix} = \begin{bmatrix} e_1 R^\epsilon_{n,T} & e_1 b_0 \\ e_1 b_0 & e_1 \end{bmatrix}. \quad \text{(IV-6)}$$

## Recursions for Rational Ladder Forms

This partitioning leads easily to the rational ladder recursions. Formally the same recursions can be used. However, the fact that the forward predictor is triangular simplifies and actually makes the recursions possible at all, since the one-step prediction errors are not needed until the next iteration, i.e. they are *feed back* and treated at the next time step as the ("other half" of the) observations. It is easily verified in the same context that half of the entries in $\Delta_{n,T}$ are zero, which guarantees that the one-step prediction errors are not used before they are required in the recursion. The following Figure 2. clearly shows this.
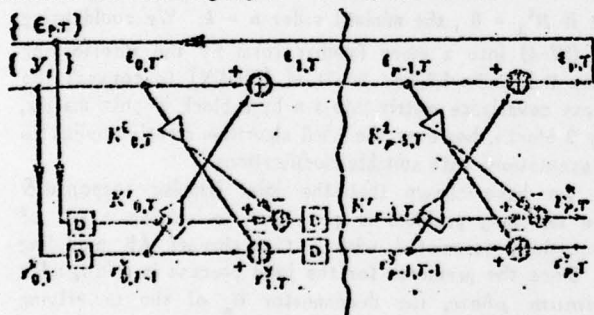
Figure 2. *Rational Ladder realization of exact one-step least-squares predictor.*

## Appendix: Computer Simulations
### Layered Media Identification

The modeling on of layered media is of interest in many areas, notably in Geophysics, see e.g. Claerbout [Cla1,2] and more recently in medical imaging or nondestructive testing. There are two basic situations that occur in these areas. The first one, where the source is on the opposite side of the receiver is the straight forward case, it leads to autoregressive or all-pole models, wich can be readily identified by using the various methods to estimate reflection coefficients by cross-correlation of the forward and backward residuals in the whitening filter in ladder form, see e.g. [Cla1,2]. The second case, where the source and receiver are on the same side did up to now not lead to such simple processing as the first case, because the (imput) transfer function is rational in general, or in the best case where a total reflection occurs within some layer the transfer function is an all-pass network. In this case the zeros are equal to the reflected poles and the 'reflection coefficients' of the numerator polynomial

are the negative of the ones of the denominator polynomial of the transfer function. We can readily see then, that our rational ladder form specializes and we get only one set of reflection coefficients that can be associated with the ones of the layered medium that generated the data. This particular case is treated from a circuit point of view by Kung [Kun]. Figure 3 shows an example using real ultrasound returns and the estimates of the reflection coefficients using a ladder structure.
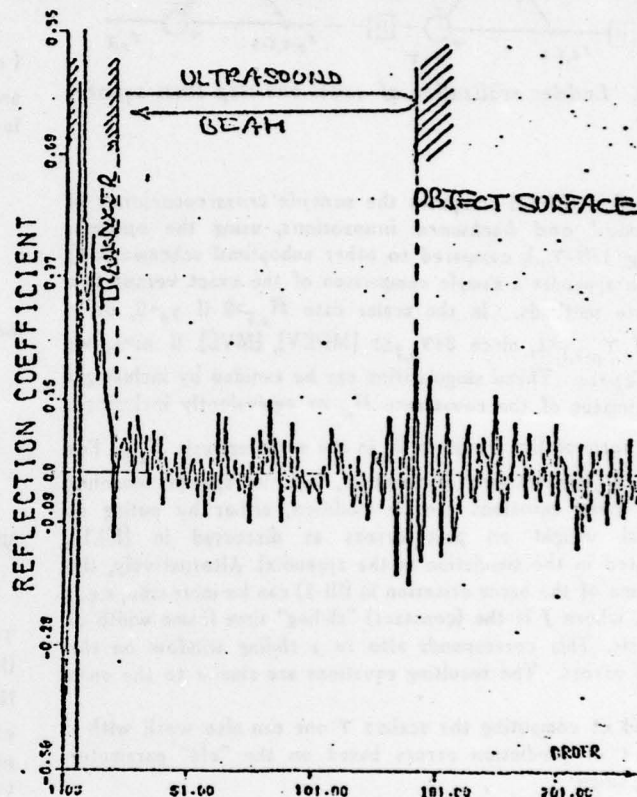
Figure 3: *Identification of a layered media via ultrasound.*

The experiments were performed by Linda Joint and Dough Boyd in the Stanford Electronics Laboratory of Prof. J. Meindl. The reflection coefficient estimates appear to be much smaller than anticipated from the experimental set up, this is due to several factors: The ladder structure actually identifies not only the medium and the single reflecting plate in the path of the ultrasound beam, but also computes an equivalent layer model for the transducer. The estimated values of the first large reflection coefficients show, that the transducer is very inefficient and not very well matched because the largest value is very close to one, which tend to "turn off" all higher order reflection coefficients. Further more, because of the wavelengs used the layers of the medium have a continuous reflection coefficient density which indicates that this direct scheme must fail since we tried to estimate the derivative of a function (with noisy data!) It would require the use of a modified ladder form that is parameterized by the equivalent of the "area function" used for instance in the speech modeling context [Wak].

## Sample Comparison of Different Reflection Coefficient Estimates

Ladder coefficient estimates were in the past said to converge very slowly, indeed this is the case for approximate recursive methods as demonstrated in the Figure 4, where three methods are compared. Two approximate recursive methods using arithmetic mean definitions of the prediction error (see e.g. [Mak], [MLVK]) and and other computationaly attractive method using the average of the product of the signs of the forward and backward

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prediction error, which arises from L1 norm considerations see Claerbout[Cla3] and has often been used in circuit design.
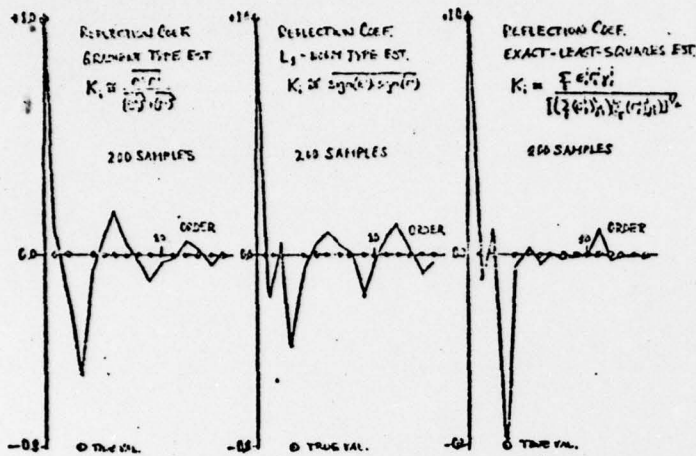


*Figure 4: Comparison of Two Approximate and One Exact Recursive Method.*

The third method uses our recursive exact least-squares equations (pre-windowed case). The first two schemes give very similar results, i.e. a bias of 50% on the only nonzero third reflection coefficient (−.8) and sidelobes as high as 20% and 10% typical value, whereas our exact method has virtualy no bias and half the maximal and typical sidelobe values. Furthermore, they actually converged already after around 30 samples compared with the 200 samples used in Figure 4 . We may note that the other schemes took much longer to actually converge.

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